Lecture 3

Special Relativity – II.

Objectives:

- *Four vectors*

Reading: *Schutz chapter 2, Rindler chapter 5, Hobson chapter 5*

3.1 The interval of SR

To cope with shifts of origin, restrict to the interval between two events

$$\Delta s^2 = (ct_2 - ct_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 ,$$

or

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2,$$

or finally with infinitesimals:

$$\boxed{ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2.} \tag{3.1}$$

ds^2 is the same in all inertial frames. It is a Lorentz scalar. Writing

$$ds^2 = c^2\, d\tau^2,$$

defines the "proper time" τ, which is the same as the coordinate time t when $dx = dy = dz = 0$. i.e. proper time is the time measured on a clock travelling with an object.

Introducing $x^0 = ct$, etc again, we can write

$$\boxed{ds^2 = c^2\, d\tau^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta,} \tag{3.2}$$

where

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.3}$$

The interval is the SR equivalent of length corresponding to the relation for lengths in Euclidean 3D

$$dl^2 = dx^2 + dy^2 + dz^2.$$

NB There is no standard sign convention for the interval and $\eta_{\alpha\beta}$. Make sure you know the convention used in textbooks.

3.2 The grain of SR

The minus signs in the definition of ds^2 means there are three types of interval:

$ds^2 > 0$ timelike intervals. Intervals between events on the wordlines of massive particles are timelike.

$ds^2 = 0$ Null intervals. Intervals between events on the wordlines of massless particles (photons) are null.

$ds^2 < 0$ Spacelike intervals which connect events out of causal contact.

These impose a distinct structure on spacetime.

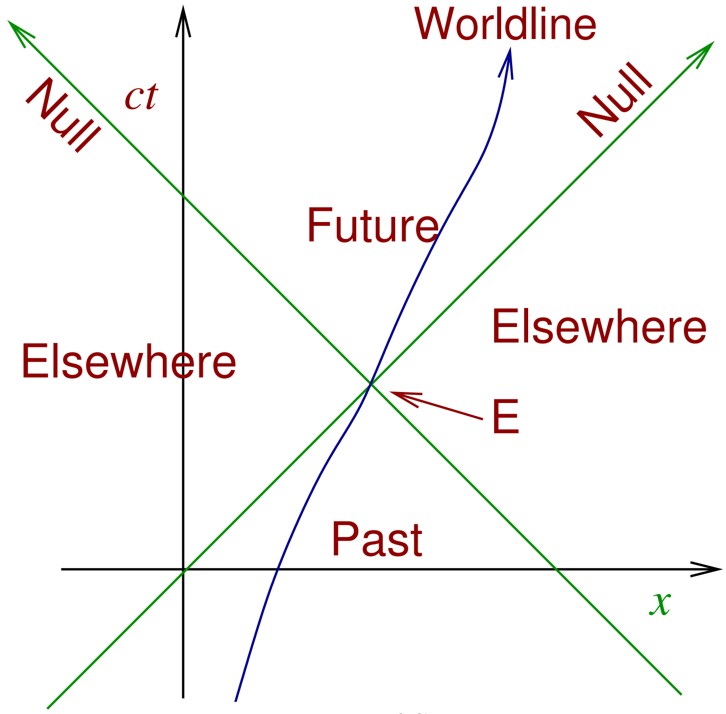

Figure: The invariant interval of SR slices up spacetime relative to an event E into past, future and "elsewhere', the latter being the events not causally connected to E.

These so-called "light-cones" are preserved in GR but are distorted according to the coordinates used.

3.3 Four-vectors

Any quantity that transforms in the same way as $\vec{X} = (x^0, x^1, x^2, x^3)$ is called a "four vector" (or often just a "vector"). Thus $\vec{V}$ is <u>defined</u> to be a vector if and only if

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta}.$$

Useful because:

- Four vectors can often be identified easily
- The way they transform follows from the LTs.
- Lead to Lorentz scalars equivalent to ds^2.

3.3.1 Four-velocity

The four-velocity is one of the most important four-vectors. Consider

$$\vec{U} = \lim_{\delta\tau \to 0} \frac{\vec{X}(\tau + \delta\tau) - \vec{X}(\tau)}{\delta\tau} = \frac{d\vec{X}}{d\tau}.$$

Since $\vec{x}$ is a four-vector and τ is a scalar, $\vec{U}$ is clearly a four-vector.

From time dilation, $d\tau = dt/\gamma$, so

$$\vec{U} = \gamma \frac{d\vec{X}}{dt} = \gamma(c, \mathbf{v}),$$

where $\mathbf{v}$ is the normal three-velocity and is shorthand for the spatial components of the four-velocity.

3.3.2 Scalars from four-vectors

If $\vec{V}$ is a four-vector, then the equivalent of the interval $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ is

$$\boxed{\vec{V} \cdot \vec{V} = |\vec{V}|^2 = \eta_{\alpha\beta} V^\alpha V^\beta} \tag{3.4}$$

This defines the invariant "length" or "modulus" of a four-vector. It is a scalar under LTs.

This relation is fundamental. Note that $|\vec{V}|^2 \neq (V^0)^2 + (V^1)^2 + (V^2)^2 + (V^3)^2$. SR and GR are not Euclidean.

Example 3.1 *Calculate the scalar equivalent to the four-velocity $\vec{U}$.*

Answer 3.1 *Long way*

$$\begin{aligned}
\eta_{\alpha\beta} U^\alpha U^\beta &= \left(U^0\right)^2 - \left(U^1\right)^2 - \left(U^2\right)^2 - \left(U^3\right)^2, \\
&= \gamma^2 \left(c^2 - v_x^2 - v_y^2 - v_z^2\right), \\
&= \gamma^2 \left(c^2 - v^2\right), \\
&= \gamma^2 \frac{c^2}{\gamma^2} = c^2.
\end{aligned}$$

Short way: since it is invariant, calculate its value in a frame for which $\mathbf{v} = 0$ *and* $\gamma = 1$, *from which immediately* $\vec{U} \cdot \vec{U} = c^2$.

$\vec{U} \cdot \vec{U} = c^2$ is an important relation. It means that $\vec{U}$ is a <u>timelike</u> four-vector.